Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 18, 2002*
Revised: *January 21, 2003*
Accepted: *January 26, 2003*

On DBI action of the non-maximally symmetric D-branes on SU(2)

Gor Sarkissian

The Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, Trieste 34014, Italy
E-mail: gor@ictp.trieste.it

Abstract: In this paper we study the non-maximally symmetric D-branes on the SU(2) group discussed in a previous article [11]. Using the two-form defined in [11] the DBI action on the branes is constructed. This action is checked for its agreement with CFT predictions. The geometry of the branes is analyzed in detail, and the singularities of branes covering the entire group are found.

Keywords: D-branes, Conformal Field Models in String Theory.

JHEP01(2003)058

 http://jhep.sissa.it/archive/papers/jhep012003058/jhep012003058.pdf

Contents

1. Introduction

During the last years progress has been made in the understanding of branes on group manifolds. In the pioneering work of Klimcik and Severa [1] additional topological conditions required for writing down a well-defined WZW action on a world-sheet with boundary were formulated. In that work also one-, two- and three-dimensional branes on SU(2) satisfying these conditions were found, but the symmetry properties of those branes remained obscure. These topological conditions were analyzed further in [2, 3].

Following these findings, in [4] maximally-symmetric solutions to these conditions, which are quantized conjugacy classes, were found. These branes were extensively studied and, by now, an almost complete understanding of them has been established: their lagrangian formulation [4, 5], the DBI action, stability properties [6] , non-commutative geometry [7, 8] etc.

Recently, using T-duality and the boundary state formalism, in [9] and [10] non-maximally symmetric branes on group manifolds and parafermionic target spaces, called B-branes, were found, preserving only part of the diagonal chiral algebra. In [11] , the vectorial and axial symmetries, obtained with algebraic methods in [10] , were checked and confirmed in the lagrangian formulation. It was also shown in [12] that boundary conditions defining B-branes on group manifolds admit axial gauging, resulting in B-branes on parafermionic target spaces. In [13] , a generalization of these methods was suggested, associating symmetry breaking branes to certain quantized chains of subgroup embeddings.

In this work we study the geometry and the DBI action of the non-maximally symmetric branes on SU(2) introduced in [9], using the lagrangian formulation suggested in [11].

In section 2 we analyze the geometry of the branes and find that, generically, they are $(\mathrm{U}(1) \times S^2)/Z_2$. For even values of k, branes covering the whole group, in the complement of a big circle, exist and coincide with one of the branes found in [1] .

In section 3, using the relation between the DBI action and the topological two-form present in the lagrangian formulation of the WZW model on a world-sheet with boundary [14] , [15] , we compute the DBI action on these branes. We show that the target space mass computation is in agreement with the CFT prediction.

In the appendix we present in detail the two-form computation.

JHEP01(2003)058

2. Geometry of the branes

It was shown in [11] that the non-maximally symmetric D-branes introduced in [9] can be represented as a group product of the T-dualized U(1) subgroup with a conjugacy class,

$$g = Lhfh^{-1}\,, \tag{2.1}$$

where $L = e^{-i\alpha\sigma_3}$, $h \in G$ and f is a constant element. In this parametrisation, the topological two-form $\omega^{(2)}$ required to trivialize the WZW three-form on the brane,

$$\omega^{\mathrm{WZ}}(g)_{|\mathrm{brane}} = d\omega^{(2)}\,, \tag{2.2}$$

can be written as

$$\omega^{(2)}(L,h) = \omega^{f}(h) - \mathrm{Tr}\left(L^{-1}dLdCC^{-1}\right), \tag{2.3}$$

where $\omega^f(h) = h^{-1}dhfh^{-1}dhf^{-1}$. Here we explore this representation for branes on the SU(2) group manifold. In particular we are interested in checking if this representation is one-to-one. Given an element g, L can be found from the condition

$$\mathrm{Tr}\left(L^{-1}g\right) = \mathrm{Tr} f\,. \tag{2.4}$$

Parameterizing g as

$$g = \begin{pmatrix} x_0 + ix_3 & ix_1 - x_2 \\ ix_1 + x_2 & x_0 - ix_3 \end{pmatrix}, \tag{2.5}$$

where $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, and taking f as $f = e^{i\hat{\psi}\sigma_3}$ we obtain from (2.4)

$$x_0 \cos\alpha - x_3 \sin\alpha = \cos\hat{\psi}\,. \tag{2.6}$$

Using now the angles θ, ϕ and $\tilde{\phi}$,

$$\begin{aligned} x_0 &= \cos\theta\cos\tilde{\phi} \\ x_1 &= \sin\theta\cos\phi \\ x_2 &= \sin\theta\sin\phi \\ x_3 &= \cos\theta\sin\tilde{\phi}\,, \end{aligned} \tag{2.7}$$

we can write (2.6) as

$$\cos\theta\cos\left(\alpha + \tilde{\phi}\right) = \cos\hat{\psi}\,. \tag{2.8}$$

We see that, given the condition $\cos^2\theta \geq \cos^2\hat{\psi}$ satisfied by the element of the brane [9] , α generically ($\cos\hat{\psi} \neq 0$) is double-valued and equals

$$\alpha_{1,2} = \pm \arccos\left(\frac{\cos\hat{\psi}}{\cos\theta}\right) - \tilde{\phi}\,. \tag{2.9}$$

Consequently every element of the brane can be written in two ways,

$$g = L_1C_1 = L_2C_2\,, \tag{2.10}$$

where

$$L_{1,2} = e^{-i\alpha_{1,2}\sigma_3} = \begin{pmatrix} \frac{\cos\hat{\psi} \mp iA}{\cos\theta} e^{i\tilde{\phi}} & 0 \\ 0 & \frac{\cos\hat{\psi} \pm iA}{\cos\theta} e^{-i\tilde{\phi}} \end{pmatrix}, \tag{2.11}$$

$$C_{1,2} = \begin{pmatrix} \cos\hat{\psi} \pm iA & i\tan\theta(\cos\hat{\psi} \pm iA)e^{-i(\tilde{\phi}-\phi)} \\ i\tan\theta(\cos\hat{\psi} \mp iA)e^{i(\tilde{\phi}-\phi)} & \cos\hat{\psi} \mp iA \end{pmatrix} \tag{2.12}$$

and $A = \sqrt{\cos^2\theta - \cos^2\hat{\psi}}$.

It follows from this analysis that generic branes are topologically $(\mathrm{U}(1) \times S^2)/Z_2$, where Z_2 is the identification of the pairs (L_1, C_1) and (L_2, C_2). As expected, the subset invariant under this identification $\cos\hat{\psi} = \cos\theta$ is the boundary of the branes.

For even k values there exist branes, given by $\hat{\psi} = \frac{\pi}{2}$, covering the whole group. For these branes there exists the dangerous region $\cos\theta = 0$, where α is not defined. Looking at eq. (2.6) and noting that for the given values of θ and $\hat{\psi}$ the left and right sides equal zero, we see that in this region (2.6) can be satisfied for any value of α. The existence of such a region for the branes covering the whole group was actually expected, because we know that $\omega^{\mathrm{WZ}}(g)$ defines a non-trivial cohomology class and eq. (2.2) cannot be satisfied everywhere. We are led to the conclusion that the brane covering the whole group is the complement of the one-dimensional circle defined by the equation $x_0 = x_3 = 0$. It should be mentioned that this brane was already found using a different approach in [1] .

3. The DBI action

In this section we compute the DBI action on the branes using the formula [14, 15]:

$$S_{\mathrm{DBI}} = \int \sqrt{\det(G + \omega^{(2)})}\,. \tag{3.1}$$

In the appendix we show that in the coordinates (2.7) ϕ , $\tilde{\phi}$ and θ the two-form $\omega^{(2)}$ takes the form

$$\frac{\omega^{(2)}}{k\alpha'} = F_{\theta\phi} d\theta \wedge d\phi + \frac{\cos 2\theta - 1}{2} d\phi \wedge d\tilde{\phi}\,, \tag{3.2}$$

where

$$F_{\theta\phi} = \pm\frac{\tan\theta\cos\hat{\psi}}{\sqrt{\cos^2\theta - \cos^2\hat{\psi}}} = \pm\frac{2\tan\theta\cos\hat{\psi}}{\sqrt{2(\cos 2\theta - \cos 2\hat{\psi})}}\,. \tag{3.3}$$

The plus and minus signs correspond to different signs in (2.9) . Now we argue that only the positive sign should be taken. For this purpose we compute the restriction of the two-form (3.2) to the two-sphere conjugacy class given by the condition $x_0 = \cos\hat{\psi}$. This should result in the well-known topological two-form for conjugacy classes derived in [4] , [5] and [6] . It is useful to introduce coordinates γ and β on the two-sphere:

$$\begin{aligned} x_0 &= \cos\hat{\psi} \\ x_1 &= \sin\hat{\psi}\sin\gamma\cos\beta \\ x_2 &= \sin\hat{\psi}\sin\gamma\sin\beta \\ x_3 &= \sin\hat{\psi}\cos\gamma\,. \end{aligned} \tag{3.4}$$

JHEP01(2003)058

Using the formulae (2.7) we can express θ, ϕ, and $\tilde{\phi}$ as functions of γ and β. Equating $x_0^2 + x_3^2$ from (3.4) and (2.7) we obtain:

$$\cos^2\theta = \cos^2\hat{\psi} + \sin^2\hat{\psi}\cos^2\gamma\,, \tag{3.5}$$

or

$$\sin\theta = \sin\hat{\psi}\sin\gamma\,. \tag{3.6}$$

Equating x_0 from (3.4) and (2.7) and substituting (3.5) we obtain:

$$\cos\tilde{\phi} = \frac{\cos\hat{\psi}}{\sqrt{\cos^2\hat{\psi} + \sin^2\hat{\psi}\cos^2\gamma}} \tag{3.7}$$

and

$$\sin\tilde{\phi} = \frac{\sin\hat{\psi}\cos\gamma}{\sqrt{\cos^2\hat{\psi} + \sin^2\hat{\psi}\cos^2\gamma}}\,. \tag{3.8}$$

Equating x_1 and x_2 from (3.4) and (2.7) and substituting (3.6) we note that we can set

$$\phi = \beta \tag{3.9}$$

Using (3.5) ,(3.6) ,(3.7) ,(3.8) and (3.9) it is straightforward to compute

$$F_{\theta\phi} d\theta \wedge d\phi = \pm\frac{\sin\theta\cos\hat{\psi}}{\cos^2\theta} d\gamma \wedge d\beta \tag{3.10}$$

and

$$\frac{\cos 2\theta - 1}{2} d\phi \wedge d\tilde{\phi} = -\frac{\sin^3\theta\cos\hat{\psi}}{\cos^2\theta} d\gamma \wedge d\beta\,. \tag{3.11}$$

Putting together (3.10) , (3.11) , we finally get

$$\omega^{(2)}|_{x_0=\cos\hat{\psi}} = \frac{\sin\theta\cos\hat{\psi}}{\cos^2\theta}\left(\pm 1 - \sin^2\theta\right) d\gamma \wedge d\beta\,. \tag{3.12}$$

We see that taking the positive sign in (3.12) we obtain

$$\omega^{(2)}|_{x_0=\cos\hat{\psi}} = k\alpha'\sin\theta\cos\hat{\psi} d\gamma\wedge d\beta = k\alpha'\sin\gamma\sin\hat{\psi}\cos\hat{\psi} d\gamma\wedge d\beta = k\alpha'\frac{\sin\gamma\sin 2\hat{\psi}}{2} d\gamma\wedge d\beta\,, \tag{3.13}$$

which is exactly the two-form for the conjugacy class found in [4] , [5] and [6] .

We note that $F_{\theta\phi} d\theta \wedge d\phi$ coincides with the field strength found in [9] for the B-brane on the parafermion disk. (One should take into account that the radius ρ in that paper is $\rho = \sin\theta$.) This can be explained by noting that, as shown in [12] , the B-branes on the parafermion disk can be derived by the axial gauging of the B-branes on group manifolds. Fixing the gauge $\tilde{\phi} = 0$ and using Stokes' theorem we obtain the boundary term for the B-brane on the parafermion disk from the boundary term of the WZW model.

We now turn to the computation of the DBI action and mass of the branes.

In the coordinates ϕ , $\tilde{\phi}$ and θ the metric is

$$ds^2 = k\alpha' \left(d\theta^2 + \cos^2\theta \, d\tilde{\phi}^2 + \sin^2\theta \, d\phi^2\right), \tag{3.14}$$

For $\omega^{(2)}$ given by (3.2) and the metric given by (3.14) , the $G + \omega^{(2)}$ matrix is

$$G + \omega^{(2)} = k\alpha' \begin{pmatrix} 1 & F_{\theta\phi} & 0 \\ -F_{\theta\phi} & \sin^2\theta & -\sin^2\theta \\ 0 & \sin^2\theta & \cos^2\theta \end{pmatrix}. \tag{3.15}$$

Using (3.3) we compute that

$$\sqrt{\det(G + \omega^{(2)})} = \left(k\alpha'\right)^{3/2} \sqrt{\sin^2\theta + F^2_{\theta\phi}\cos^2\theta} = (k\alpha')^{3/2} \frac{\sin 2\theta}{\sqrt{2(\cos 2\theta - \cos 2\hat{\psi})}}. \tag{3.16}$$

This result is in agreement with the computation of the overlap of the boundary state with the graviton wave packet in [9]:

$$\langle B, j, \eta | \theta \rangle \sim \frac{\Theta(\cos 2\theta - \cos 2\hat{\psi})}{\sqrt{\cos 2\theta - \cos 2\hat{\psi}}}. \tag{3.17}$$

For the energy we obtain

$$E_{\rm DBI} = T_{(3)} \int_0^{\hat{\psi}} d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\tilde{\phi} \sqrt{\det(G + \omega^{(2)})} = 4\pi^2 \left(k\alpha'\right)^{3/2} T_{(3)} \sin\hat{\psi}\,. \tag{3.18}$$

Recalling now that the masses of A-branes wrapping conjugacy classes are equal to

$$M\left(A, \hat{\psi}\right) = 4\pi k\alpha' T_{(2)} \sin\hat{\psi} \tag{3.19}$$

and the relation between the tensions of the D2- and D3-branes,

$$\pi\sqrt{\alpha'}T_{(3)} = T_{(2)}\,, \tag{3.20}$$

we get

$$M\left(B, \hat{\psi}\right) = \sqrt{k} M\left(A, \hat{\psi}\right) \tag{3.21}$$

as predicted by the CFT computation.

Acknowledgments

The author thanks C. Bachas, M. Blau, K. S. Narain and C. Schweigert for useful discussions.

A. The $\omega^{(2)}$ computation

Here we use the Euler angles χ , φ and $\tilde\theta$ connected to the coordinates (2.7) by the formulae

$$\begin{aligned} \chi &= \tilde\phi + \phi \\ \varphi &= \tilde\phi - \phi \\ \tilde\theta &= 2\theta\,. \end{aligned} \tag{A.1}$$

In terms of Euler angles, g has the form

$$g = \begin{pmatrix} \cos\frac{\tilde\theta}{2} e^{i\frac{\chi+\varphi}{2}} & i\sin\frac{\tilde\theta}{2} e^{i\frac{\chi-\varphi}{2}} \\ i\sin\frac{\tilde\theta}{2} e^{-i\frac{\chi-\varphi}{2}} & \cos\frac{\tilde\theta}{2} e^{-i\frac{\chi+\varphi}{2}} \end{pmatrix}. \tag{A.2}$$

Parametrizing g as $g = LC = Lhfh^{-1}$, where $f = e^{i\hat\psi\sigma_3}$, we have

$$\omega^{(2)}(L,h) = \omega^f(h) - \mathrm{Tr}\left(L^{-1}dLdCC^{-1}\right). \tag{A.3}$$

Using (2.11) and (2.12) we compute

$$L^{-1}_{1,2}dL_{1,2} = \begin{pmatrix} \pm\frac{i\cos\hat\psi\tan\frac{\tilde\theta}{2}}{2A}d\tilde\theta + i\frac{d\chi+d\varphi}{2} & 0 \\ 0 & \mp i\frac{\cos\hat\psi\tan\frac{\tilde\theta}{2}}{2A}d\tilde\theta - i\frac{d\chi+d\varphi}{2} \end{pmatrix}, \tag{A.4}$$

$$dC_{1,2}C^{-1}_{1,2} = \begin{pmatrix} \mp i\frac{\cos\hat\psi\tan\frac{\tilde\theta}{2}}{2A}d\tilde\theta - i\sin^2\frac{\tilde\theta}{2}d\varphi & *** \\ *** & \pm i\frac{\cos\hat\psi\tan\frac{\tilde\theta}{2}}{2A}d\tilde\theta + i\sin^2\frac{\tilde\theta}{2}d\varphi \end{pmatrix}. \tag{A.5}$$

The terms denoted by stars are not important for our purposes.

Collecting (A.4) and (A.5) we obtain

$$\mathrm{Tr}\left(L^{-1}_{1,2}dL_{1,2}dC_{1,2}C^{-1}_{1,2}\right) = \pm\frac{\cos\hat\psi\tan\frac{\tilde\theta}{2}}{2A}d\tilde\theta\wedge d\chi \mp \frac{\cos\hat\psi\tan\frac{\tilde\theta}{2}\cos\tilde\theta}{2A}d\tilde\theta\wedge d\varphi + \sin^2\frac{\tilde\theta}{2}d\chi\wedge d\varphi\,. \tag{A.6}$$

Now we turn to the computation of $\omega^f(h)$.

Let us take C in the form:

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \tag{A.7}$$

where

$$\begin{aligned} c_{11} &= \cos\hat\psi + ix_3 \\ c_{12} &= ix_1 - x_2 \\ c_{21} &= ix_1 + x_2 \\ c_{22} &= \cos\hat\psi - ix_3\,, \end{aligned} \tag{A.8}$$

and $x_1^2 + x_2^2 + x_3^2 = \sin^2\hat\psi$. Now we should find an h satisfying the relation $C = hfh^{-1}$, where $f = e^{i\hat\psi\sigma_3}$. It is easy to check that it is possible to choose an h of the form

$$h = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{A.9}$$

where

$$a = -\frac{x_1 + ix_2}{\sqrt{2\sin\hat{\psi}(\sin\hat{\psi} - x_3)}}$$
$$b = \sqrt{\frac{\sin\hat{\psi} - x_3}{2\sin\hat{\psi}}}\,. \tag{A.10}$$

Substituting (A.9) in $\omega^f(h)$ we obtain

$$\omega^f(h) = 2i\sin 2\hat{\psi}(a^*b^*da\wedge db - bb^*da\wedge da^* - aa^*db^*\wedge db + abdb^*\wedge da^*)\,. \tag{A.11}$$

Substituting (A.10) in (A.11) we get:

$$\omega^f(h) = \frac{dc_{12}\wedge dc_{21}}{ix_3}\cos\hat{\psi}\,. \tag{A.12}$$

Finally, using (2.12) we get

$$\omega^f(h) = \mp\frac{\tan\frac{\tilde{\theta}}{2}\cos^2\frac{\tilde{\theta}}{2}\cos\hat{\psi}}{\sqrt{\cos^2\frac{\tilde{\theta}}{2} - \cos^2\hat{\psi}}}d\tilde{\theta}\wedge d\varphi\,. \tag{A.13}$$

Inserting now (A.6) and (A.13) in (A.3) we obtain:

$$\omega^2(L,h) = \pm\frac{\cos\hat{\psi}\tan\frac{\tilde{\theta}}{2}}{2\sqrt{\cos^2\frac{\tilde{\theta}}{2}\mp\cos^2\hat{\psi}}}d\tilde{\theta}\wedge d\chi\mp\frac{\cos\hat{\psi}\tan\frac{\tilde{\theta}}{2}}{2\sqrt{\cos^2\frac{\tilde{\theta}}{2}-\cos^2\hat{\psi}}}d\tilde{\theta}\wedge d\varphi - \sin^2\frac{\tilde{\theta}}{2}d\chi\wedge d\varphi\,. \tag{A.14}$$

Then, using the coordinates (2.7) we can rewrite (A.14) compactly in the form

$$\frac{\omega^2(L,h)}{2} = \pm\frac{2\cos\hat{\psi}\tan\theta}{\sqrt{2(\cos 2\theta - \cos 2\hat{\psi})}}d\theta\wedge d\phi - \sin^2\theta d\phi\wedge d\tilde{\phi}\,. \tag{A.15}$$

It follows from conformal invariance that in the DBI action we should use half of $\omega^2(L,h)$ as in (A.15) .

References

[1] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112].

[2] J.M. Figueroa-O'Farrill and S. Stanciu, *D-brane charge, flux quantization and relative (co)homology*, *J. High Energy Phys.* **01** (2001) 006 [hep-th/0008038].

[3] K. Gawędzki and N. Reis, *WZW branes and gerbes*, hep-th/0205233.

[4] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[5] K. Gawędzki, *Conformal field theory: a case study*, in *Conformal Field Theory*, Y. Nutku, C. Saclioglu, T. Turgut eds., Frontier in Physics **102**, Perseus Publishing, 2000, [hep-th/9904145].

[6] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[7] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040].

[8] A.Y. Alekseev, S. Fredenhagen, T. Quella and V. Schomerus, *Non-commutative gauge theory of twisted D-branes*, *Nucl. Phys.* **B 646** (2002) 127 [hep-th/0205123].

[9] J.M. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *J. High Energy Phys.* **07** (2001) 046 [hep-th/0105038].

[10] J.M. Maldacena, G.W. Moore and N. Seiberg, *D-brane instantons and K-theory charges*, *J. High Energy Phys.* **11** (2001) 062 [hep-th/0108100].

[11] G. Sarkissian, *Non-maximally symmetric D-branes on group manifold in the lagrangian approach*, *J. High Energy Phys.* **07** (2002) 033 [hep-th/0205097].

[12] M.A. Walton and J.-G. Zhou, *D-branes in asymmetrically gauged WZW models and axial-vector duality*, *Nucl. Phys.* **B 648** (2003) 523 [hep-th/0205161].

[13] T. Quella, *On the hierarchy of symmetry breaking D-branes in group manifolds*, *J. High Energy Phys.* **12** (2002) 009 [hep-th/0209157].

[14] P. Bordalo, S. Ribault and C. Schweigert, *Flux stabilization in compact groups*, *J. High Energy Phys.* **10** (2001) 036 [hep-th/0108201].

[15] S. Stanciu, *A note on D-branes in group manifolds: flux quantization and D0-charge*, *J. High Energy Phys.* **10** (2000) 015 [hep-th/0006145].

JHEP01(2003)058